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Scattering of a surface wave by a submerged sphere ⋆

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SUMMARY
The problem of the scattering of a surface wave in a nonviscous, incompressible fluid of infinite depth by a fully submerged, rigid, stationary sphere has been reduced to the solution of an infinite set of linear algebraic equations for the expansion coefficients in spherical harmonics of the velocity potential. These equations are easily solved numerically, so long as the sphere is not too close to the surface. The approach has been to formulate the problem as an integral equation, expand the Green's function, the velocity potential of the incident wave, and the total velocity potential in spherical harmonics, impose the boundary condition at the surface of the sphere, and carry out the integrations. The scattering cross section has been evaluated numerically and is shown to peak for values of the product of radius and wave number somewhat less than unity. Also, the Born approximation to the cross section is obtained in closed form.

1. Introduction

The problem of determining the influence of a submerged body on the ambient surface wave structure is an old one, going back at least to Thomson [1] and Lamb [2]. But while a great deal of effort has been devoted since then to the solution of various special cases and the development of a variety of approximation methods (see Refs. [3]–[10] and [24] for an incomplete but representative sample), one of the simplest and most important cases — the modification (i.e., scattering) of a surface wave by a completely submerged, rigid, stationary sphere — has remained unsolved.

The present work provides an exact solution to this problem — though not in closed form — for a homogeneous, incompressible, nonviscous fluid of infinite depth and infinite extent. The approach is to cast the equation for the velocity potential into the form of an integral equation involving the Green's function; an explicit expression for this function was obtained long ago in Kochin's [3] pioneering work, rederived independently by John [5], and further developed by Wehausen and Laitone [11]. Expansion in spherical harmonics of the Green's function and the velocity potentials of the known incident and the unknown modified wave permits analytic evaluation of the integrals and leads to an infinite set of linear algebraic equations for the unknown expansion coefficients. So long as the spherical obstacle is not too close to the surface, only the first few harmonics contribute significantly, and the corresponding coefficients are readily found by truncating the expansion. The equation of the free surface, which embodies the desired modification of the incident wave, is

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obtained directly from this modified velocity potential. This approach makes it very easy to calculate both the total and the partial scattering cross sections of such a submerged sphere.

The Born approximation to the scattering cross section has been calculated. It can be evaluated in closed form as a very simple expression involving modified Bessel functions, and is shown to be an excellent approximation to the exact scattering cross section whenever the depth of the sphere center exceeds the diameter. The Born approximation is used to interpret physically the qualitative features of the dependence of the cross section on the two dimensionless parameters: the ratio of the depth of the sphere center to the radius, a, and the ratio of a to the surface wavelength.

The structure of the paper is as follows. In Section 2, the problem is formulated as an integral equation with a Green's function kernel. In Section 3, this integral equation is used to derive an infinite set of linear algebraic equations for the coefficients in a spherical-harmonic expansion of the velocity potential. In Section 4, the scattering cross section and the Born approximation are obtained. In Section 5, a version of the Optical Theorem relevant to surface-wave scattering is derived. In Section 6, the physical significance of the results is discussed.

2. Formulation of the problem

Suppose a rigid sphere of radius a is submerged in an unbounded fluid of infinite depth, such that its center is at a distance d ($> a$) below the surface. We use a Cartesian coordinate system with its origin at the center of the sphere, with the axes directed such that the fluid surface corresponds to the plane $z = d$. (See Fig. 1) The velocity potential $\Phi(x, y, z, t)$ is specified completely by the incompressible nonviscous fluid equation

$$\nabla^2\Phi = 0 \text{ for } z \le d, r \ge a, \tag{1a}$$

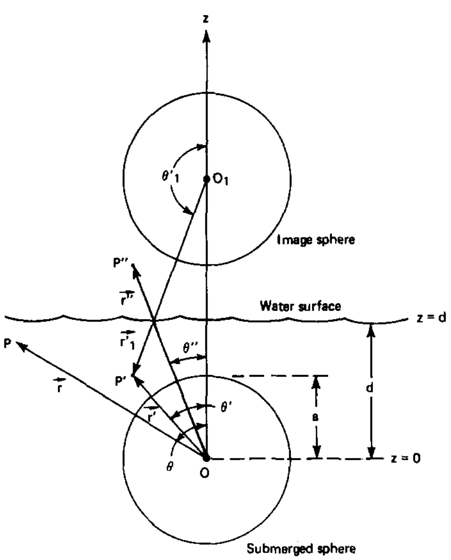

Figure 1. Geometry of submerged sphere and image sphere.

the free surface boundary condition

$$\frac{\partial \Phi}{\partial z} = -\frac{1}{g}\frac{\partial^2 \Phi}{\partial t^2} \text{ for } z = d, \tag{1b}$$

where g is the acceleration due to gravity, the "bottom" boundary condition

$$\frac{\partial \Phi}{\partial z} \to 0 \text{ for } z \to -\infty, \tag{1c}$$

and the boundary condition on the surface of the obstacle, where the normal derivative of Φ must vanish,

$$\frac{\partial \Phi}{\partial n} = 0 \text{ for } r \equiv (x^2 + y^2 + z^2)^{\frac{1}{2}} = a. \tag{1d}$$

Suppose we consider waves having a single frequency $\omega/2\pi$. Let us define the (complex) spatial part ψ of the velocity potential through the relation

$$\Phi(x, y, z, t) = \mathrm{Re}\{\psi(x, y, z)\mathrm{e}^{-i\omega t}\}. \tag{2}$$

The equation and boundary conditions for ψ are given by

$$\nabla^2 \psi = 0 \text{ for } z \leq d,\ r \geq a, \tag{3a}$$

$$\frac{\partial \psi}{\partial z} = \nu\psi \text{ for } z = d, \text{ where } \nu = \frac{\omega^2}{g}, \tag{3b}$$

$$\frac{\partial \psi}{\partial z} \to 0 \text{ for } z \to -\infty, \tag{3c}$$

and

$$\frac{\partial \psi}{\partial r} = 0 \text{ for } r = a. \tag{3d}$$

We solve this equation in terms of a Green's function,

$$\psi(\boldsymbol{r}) = \psi_i(\boldsymbol{r}) + 1/4\pi \int_s \psi(\boldsymbol{r}')\nabla' G(\boldsymbol{r}, \boldsymbol{r}')\cdot \boldsymbol{n}'\, d^2 s', \tag{4}$$

where s is the surface of the spherical obstacle ($r = a$), $\boldsymbol{r}'$ is the vector from 0 to P' (Fig. 1), ∇' denotes the gradient with respect to the "primed" variable, $\boldsymbol{n}$ is the unit normal from the surface into the fluid and ψ_i is the "incident" (i.e., unscattered) free surface wave of unit amplitude, which we assume to be propagating in the x-direction,

$$\psi_i = \mathrm{e}^{\nu(z-d)+i\nu x}. \tag{5}$$

In order that the solution to Eq. (4) be identical to that of Eqs. (3a)–(3d), the Green's function, G, for a point source at $\boldsymbol{r}'$ must satisfy the equation

$$\nabla^2 G = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{6a}$$

and the same "top" and "bottom" boundary conditions as ψ, namely

$$\frac{\partial G}{\partial z} = \nu G \text{ for } z = d, \tag{6b}$$

and

$$\frac{\partial G}{\partial z} \to 0 \text{ for } z \to -\infty; \tag{6c}$$

the boundary condition satisfied by ψ on the surface of the obstacle, Eq. (3d), must be imposed separately.

Wehausen and Laitone [11] have shown that G, as determined by Eqs. (6a)–(6c), has the form

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} + \frac{1}{|\boldsymbol{r} - \boldsymbol{r}''|} + 2\nu \,⨍_0^\infty e^{k(z+z'-2d)} J_0(kR) \frac{dk}{k - \nu}$$
$$+ 2\pi i \nu e^{\nu(z+z'-2d)} J_0(\nu R), \tag{7}$$

where $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{r}' = (x', y', z')$, $R = [(x - x')^2 + (y - y')^2]^{\frac{1}{2}}$, $\boldsymbol{r}'' = (x', y', 2d - z')$, which is the vector from 0 to P'', the reflection of P' in the free surface (see Fig. 1), $⨍$ denotes the principal value, and J_0 is the Bessel function of order zero.

3. Calculation of velocity potential

We now proceed to solve Eq. (4) for the velocity potential. To that end, we expand ψ, ψ_i and G in spherical harmonics centered at the origin. The expansion coefficients of ψ will be the unknown quantities that specify the solution. After imposing the boundary condition at the surface of the sphere, Eq. (3d), and carrying out all the integrals in Eq. (4), we shall be left with an infinite set of linear equations for the desired coefficients.

The general solution of the incompressible fluid equation $\nabla^2\psi = 0$, expressed in spherical coordinates, has the form

$$\psi = \sum_{m,n} \varepsilon_m (F_{mn} r^n + G_{mn} r^{-n-1}) P_n^m(\cos\theta) \cos m\varphi \tag{8}$$

where

$$\varepsilon_m = \begin{cases} 1 \text{ if } m = 0 \\ 2 \textit{ otherwise,} \end{cases}$$

the summation $\sum_{m,n}$ denotes $\sum_{m=0}^\infty \sum_{n=m}^\infty$, P_n^m is the associated Legendre polynomial, and (θ, φ) are the spherical angular coordinates of $\boldsymbol{r}$ with respect to the center of the spherical obstacle, and with the z-axis chosen as the polar direction. Imposition of the boundary condition $\partial\psi/\partial r = 0$ for $r = a$ gives the relationship

$$G_{mn} = \left(\frac{n}{n+1}\right) a^{2n+1} F_{mn}. \tag{8a}$$

Imposition of the same boundary condition in Eq. (4) leads to

$$\left(\frac{\partial\psi_i}{\partial r}\right)_{r=a} + 1/4\pi \int_s \psi(\mathbf{r}')\left[\frac{\partial^2}{\partial r\,\partial r'} G(\mathbf{r}, \mathbf{r}')\right]_{r=a, r'=a} d^2s' = 0. \tag{9}$$

Expansion of the incident wave, ψ_i, and of the various terms of the Green's function is carried out in Appendix A:

$$2\pi i\nu \mathrm{e}^{\nu(z+z'-2d)} J_0(\nu R) = \sum_{m,n,\kappa} \varepsilon_m A_{mn\kappa} P_n^m(\cos\theta) P_\kappa^m(\cos\theta') \cos m(\varphi - \varphi'), \tag{10a}$$

$$2\nu \int_0^\infty \frac{dk}{k-\nu} \mathrm{e}^{k(z+z'-2\nu)} J_0(kR) = \sum_{m,n,\kappa} \varepsilon_m B_{mn\kappa} P_n^m(\cos\theta) \mathrm{P}_\kappa^m(\cos\theta') \cos m(\varphi - \varphi'), \tag{10b}$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{m,n} \varepsilon_m C_{mn} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\varphi - \varphi'), \tag{10c}$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}''|} = \sum_{m,n,\kappa} \varepsilon_m D_{mn\kappa} P_n^m(\cos\theta) P_\kappa^m(\cos\theta') \cos m(\varphi - \varphi'), \tag{10d}$$

$$\psi_i = \sum_{m,n} \varepsilon_m E_{mn} P_n^m(\cos\theta) \cos m\varphi, \tag{10e}$$

where the summation $\sum\limits_{m,n,\kappa}$ denotes $\sum\limits_{m=0}^{\infty} \sum\limits_{n=m}^{\infty} \sum\limits_{\kappa=m}^{\infty}$,

$$A_{mn\kappa} = \frac{2\pi i\nu \mathrm{e}^{-2\nu d}(-1)^m (\nu r)^n (\nu r')^\kappa}{(n+m)!\,(\kappa+m)!}, \tag{11a}$$

$$B_{mn\kappa} = \frac{2\nu(-1)^{m+n+\kappa+1}}{(n+m)!\,(\kappa+m)!} (\nu r)^n (\nu r')^\kappa \frac{\partial^{n+\kappa}}{\partial y^{n+\kappa}} [\mathrm{e}^{-y} \overline{Ei}(y)]\Big|_{y=2\nu d}, \tag{11b}$$

[where $\overline{Ei}(y)$ is the exponential integral defined by

$$\overline{Ei}(y) = -\int_{-y}^{\infty} \mathrm{e}^{-t} \frac{dt}{t} \quad \text{for } y > 0]$$

which can be simplified to

$$B_{mn\kappa} = \frac{2\nu(-1)^{m+1}(\nu r)^n(\nu r')^\kappa}{(n+m)!\,(\kappa+m)!} \left[\mathrm{e}^{-2\nu d}\overline{Ei}(2\nu d) - \sum_{j=0}^{n+\kappa-1} \frac{j!}{(2\nu d)^{j+1}}\right], \tag{11b'}$$

$$C_{mn} = \frac{r'^n}{r^{n+1}} \frac{(n-m)!}{(n+m)!} \quad \text{for } r' \le r, \tag{11c}$$

$$D_{mn\kappa} = \frac{1}{2d}\left(\frac{r}{2d}\right)^n \left(\frac{r'}{2d}\right)^\kappa \frac{(n+\kappa)!}{(n+m)!\,(\kappa+m)!}, \tag{11d}$$

and

$$E_{mn} = \frac{i^m (vr)^n}{(m+n)!} \mathrm{e}^{-vd}. \tag{11e}$$

Due to the orthogonality of the spherical harmonics, Eq. (9) is separately valid for each (m, n) term in the spherical harmonic expansion. Thus we have

$$\frac{\partial E_{mn}(a)}{\partial r} = -\frac{a^2}{4\pi}\int_0^{2\pi} d\varphi' \cos m\varphi' \int_0^{\pi} d\theta' \sin\theta' \psi(a, \theta', \varphi') \frac{\partial^2 Q_{mn}(\theta', a, a)}{\partial r \partial r'}$$

$$\text{for } 0 \le m \le n,\ n = 0, 1, 2, \ldots \tag{12}$$

where

$$Q_{mn}(\theta', r, r') = \sum_{\kappa=m}^{\infty} (C_{mn}\delta_{n\kappa} + A_{mn\kappa} + B_{mn\kappa} + D_{mn\kappa}) P_\kappa^m(\cos\theta'), \tag{13}$$

with

$$\delta_{n\kappa} = \begin{cases} 1 & \text{if } n = \kappa \\ 0 & \text{otherwise.} \end{cases}$$

Using Eqs. (8) and (8a) for ψ, and carrying out the angular integrations indicated above, we find

$$e_{mn} = \sum_{\kappa=m}^{\infty} \frac{f_{m\kappa}}{\kappa+1} \frac{(\kappa+m)!}{(\kappa-m)!} (c_{mn}\delta_{n\kappa} + a_{mn\kappa} + b_{mn\kappa} + d_{mn\kappa}) \qquad \text{for } 0 \le m \le n, \tag{14}$$

where

$$a_{mn\kappa} = a^3 \left.\frac{\partial^2 A_{mn\kappa}}{\partial r \partial r'}\right|_{r=r'=a} = \frac{2\pi i(-1)^m n\kappa (va)^{n+\kappa+1} \mathrm{e}^{-2vd}}{(n+m)!\,(\kappa+m)!}, \tag{15a}$$

$$b_{mn\kappa} = a^3 \left.\frac{\partial^2 B_{mn\kappa}}{\partial r \partial r'}\right|_{r=r'=a} = \frac{2(-1)^{m+1} n\kappa (va)^{n+\kappa+1}}{(n+m)!\,(\kappa+m)!}$$

$$\times \left[\mathrm{e}^{-2vd}\,\overline{Ei}(2vd) - \sum_{j=0}^{n+\kappa-1} \frac{j!}{(2vd)^{j+1}}\right], \tag{15b}$$

$$c_{mn} = a^3 \left.\frac{\partial^2 C_{mn}}{\partial r \partial r'}\right|_{r=r'=a} = -n(n+1)\frac{(n-m)!}{(n+m)!}, \tag{15c}$$

$$d_{mn\kappa} = a^3 \left.\frac{\partial^2 D_{mn\kappa}}{\partial r \partial r'}\right|_{r=r'=a} = \frac{n\kappa(n+\kappa)!}{(n+m)!\,(\kappa+m)!}\left(\frac{a}{2d}\right)^{n+\kappa+1}, \tag{15d}$$

$$e_{mn} = a\left.\frac{\partial E_{mn}}{\partial r}\right|_{r=a} = \frac{i^m (va)^n n}{(n+m)!}\,e^{-vd}, \tag{15e}$$

and

$$f_{mn} = a^n F_{mn}. \tag{15f}$$

Eq. (14) is an infinite set of linear algebraic equations for the unknown expansion coefficients $f_{m\kappa}$ which specify the spatial part ψ of the velocity potential Φ. Substitution of Eqs. (15a)–(15e) transforms this set of equations into the equivalent set

$$-(n+m)!f_{mn} + \sum_{\kappa=m}^{\infty} \frac{f_{m\kappa}\kappa\alpha^{n+\kappa+1}}{(\kappa+1)(\kappa-m)!} \times \left\{(n+\kappa)! + 2\beta^{n+\kappa+1}\left[e^{-\beta}(\pi i - \overline{Ei}(\beta)) + \sum_{j=0}^{n+\kappa-1}\frac{j!}{\beta^{j+1}}\right]\right\} = -i^m(\alpha\beta)^n e^{-\beta/2}$$

$$\text{for } 0 \le m \le n,\ n = 0, 1, 2, \ldots \tag{16}$$

where $\alpha = a/(2d)$ and $\beta = 2vd$.

The spatial part of the velocity potential is expressed in terms of the f_{mn} as

$$\psi(r, \theta, \varphi) = \sum_{m,n} \varepsilon_m \left[\left(\frac{r}{a}\right)^n + \left(\frac{n}{n+1}\right)\left(\frac{a}{r}\right)^{n+1}\right] f_{mn} \cos m\varphi\, P_n^m(\cos\theta). \tag{17}$$

The equation of the surface η is expressed in terms of ψ by

$$\eta(x, y, t) = -\frac{1}{g}\frac{\partial\Phi}{\partial t}(x, y, d, t) = \mathrm{Re}\left\{\frac{i\omega}{g} e^{-i\omega t}\psi(x, y, d)\right\}$$

$$= \frac{\omega}{g}\left[\sin\omega t\,\psi^r(x, y, d) - \cos\omega t\,\psi^i(x, y, d)\right], \tag{18}$$

where ψ^r and ψ^i are the real and imaginary part of ψ, respectively.

For the cross section calculations in Section 4 we shall require the asymptotic behavior at large distances from the obstacle, but at moderate depths, of the scattered part of the velocity potential, $\Psi_s = \Psi - \Psi_i$. This has been evaluated in Appendix B, with the result

$$\psi_s \sim (8\pi/v\rho)^{\frac{1}{2}} e^{v(z-2d)} e^{iv\rho} e^{i\pi/4} \sum_{m,n} \varepsilon_m i^{-m} \frac{(va)^{n+1} n f_{mn}}{(n-m)!\,(n+1)} \cos m\varphi, \tag{19}$$

where $\rho = (x^2 + y^2)^{\frac{1}{2}}$.

4. Cross-section calculations: exact and Born approximation

We adopt the usual definition for the total scattering cross section, σ, of the obstacle, but adapted to a *surface* wave, namely,

$$\sigma = P_s/I_i, \tag{20}$$

where P_s is the total power in the scattered wave and I_i is the intensity (power per unit *length* along the wavefront) of the incident wave. The intensity can be obtained by evaluating

$$I_i = \int_{-\infty}^{d} \langle p_i v_i \rangle dz, \tag{21}$$

where p_i is the component of pressure at radian frequency ω for the incident wave, v_i is the horizontal component of the fluid particle velocity for that wave, and the brackets $\langle\ \rangle$ denote the time average. These quantities can both be expressed in terms of the velocity potential ψ_i:

$$p_i = \mathrm{Re}(-i\omega\rho_f\psi_i \mathrm{e}^{-i\omega t}) \tag{22}$$

and

$$v_i = \mathrm{Re}\left(-\frac{\partial\psi_i}{\partial x}\mathrm{e}^{-i\omega t}\right), \tag{23}$$

where ρ_f is the fluid density. The scattered power is similarly given by

$$P_s = \rho\int_0^{2\pi} d\varphi \int_{-\infty}^{d} \langle p_s v_s \rangle dz, \tag{24}$$

with

$$p_s = \mathrm{Re}(-i\omega\rho_f\psi_s \mathrm{e}^{-i\omega t}) \tag{25}$$

and

$$v_s = \mathrm{Re}\left(-\frac{\partial\psi_s}{\partial \rho}\mathrm{e}^{-i\omega t}\right). \tag{26}$$

For lossless wave propagation, P_s is of course independent of ρ, and may therefore be evaluated in the asymptotic region, i.e., for large ρ. Using the asymptotic approximation, Eq. (19), for ψ_s gives for the cross section

$$\sigma = \frac{2\pi}{\nu}\mathrm{e}^{-4\nu d}\sum_{m=0}^{\infty} \varepsilon_m |A_m|^2, \tag{27}$$

where the A_m are proportional to the Fourier expansion coefficients of the asymptotic form for ψ_s evaluated at the surface, and are defined by

$$\psi_s(\mathrm{surf}) \sim \frac{\mathrm{e}^{i\nu\rho}}{(\nu\rho)^{\frac{1}{2}}}\mathrm{e}^{-2\nu d}\sum_{m=0}^{\infty}\varepsilon_m A_m \cos m\varphi. \tag{28}$$

They are therefore given by

$$A_m = (8\pi)^{\frac{1}{2}} i^{-m} \mathrm{e}^{i\pi/4}\sum_{n=m}^{\infty}\frac{(\nu a)^{n+1} n\bar{f}_{mn}}{(n-m)!\,(n+1)}, \tag{29}$$

where we have set $f_{mn} = e^{-\nu d}\bar{f}_{mn}$ in order to display explicitly the exponential $e^{-\nu d}$ dependence introduced into the f_{mn} by the factor $e^{-\beta/2}$ on the right-hand side of Eq. (16).

It is instructive to calculate the Born approximation to the scattering cross section, σ_B. This is defined as

$$\sigma_B = P_s^B/I_i, \tag{30}$$

where P_s^B is defined as in Eqs. (24)–(26), but with the exact scattered part of the velocity potential, ψ_s, replaced by the Born approximation, ψ_s^B, given by

$$\psi_s^B(\boldsymbol{r}) = 1/4\pi \int_s \psi_i(\boldsymbol{r}')\nabla' G(\boldsymbol{r}, \boldsymbol{r}')\cdot \boldsymbol{n}' d^2 s'. \tag{31}$$

This is similar to Eq. (4), but with ψ replaced by ψ_i in the integrand. Such an approximation should be useful when the velocity potential on the surface of the sphere is not greatly modified by the scattering process, as happens when the sphere is not too close to the surface.

Only the asymptotic form of ψ_s^B is needed for the cross section calculation, and this has been calculated in Appendix B, with the result

$$\psi_s^B \sim (\psi_s^B)_{\text{asymp}} = (8\pi/\nu\rho)^{\frac{1}{2}} e^{\nu(z-3d)} e^{i\nu\rho} e^{i\pi/4} (\nu a)^2 \cos\frac{\varphi}{2}\, i_1\!\left(2\nu a \cos\frac{\varphi}{2}\right), \tag{32}$$

where i_1 is the modified spherical Bessel function of the first kind of order 1, which can be expressed in terms of elementary functions,

$$i_1(x) = (\pi/2x)^{\frac{1}{2}} I_{\frac{3}{2}}(x) = \frac{\cosh x}{x} - \frac{\sinh x}{x^2}; \tag{33}$$

here $I_{\frac{3}{2}}$ is the modified Bessel function of the first kind of order $\frac{3}{2}$. The Born approximation to the normalized cross section is then given in terms of this asymptotic form as

$$\frac{\sigma_B}{2a} = \frac{\nu\rho}{a}\int_{-\infty}^{d} dz \int_0^{2\pi} |(\psi_s^B)_{\text{asymp}}|^2 d\varphi. \tag{34}$$

This integral has been calculated in Appendix C, with the result

$$\frac{\sigma_B}{2a} = \pi^2 e^{-4\nu d}\{\nu a[I_0(4\nu a) + 1] - I_1(4\nu a)\}. \tag{35}$$

Fig. 2 shows plots of $\sigma/2a$ and $\sigma_B/2a$ as functions of νa, for two values of the parameter d/a. For the more shallow sphere, with $d/a = 1.5$, the Born approximation gives a better estimate of the maximum value of the cross section, but somewhat poorer estimates of the location of that maximum and of the width of the curve.

These same plots for σ are given in Fig. 3 on a semi-log scale, to show that there are some

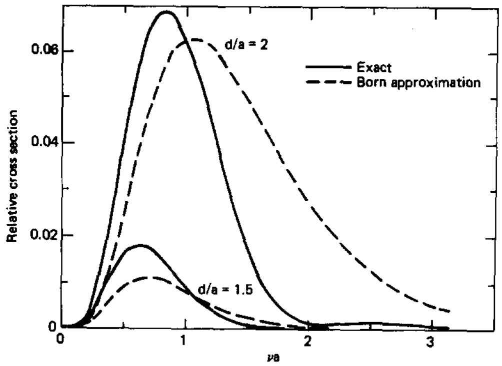

Figure 2. Exact normalized scattering cross section and Born approximation *vs.* va.

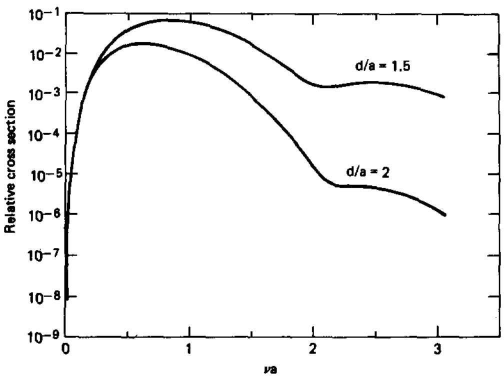

Figure 3. Exact normalized cross section *vs.* va.

secondary peaks for larger values of va. These occur because the various partial cross sections, σ_m,

$$\sigma_m = \frac{2\pi}{v} e^{-4vd} \varepsilon_m |A_m|^2 \tag{36}$$

have diffraction sidelobes for large va. This is illustrated in Figs. 4a and 4b, which give plots of the first four normalized partial cross sections, $\sigma_m/2a$, $m = 0, 1, 2, 3$. Note that the diffraction structure is greatly attenuated as compared with such structure in more familiar scattering problems such as Mie scattering of electromagnetic or scalar waves by a sphere. This point is further elucidated in Section 6.

5. Application of the Optical Theorem

How many terms are needed in Eq. (27) for an accurate cross section calculation can be determined by applying the well-known "Optical Theorem" ([12]). This theorem is a mathematical expression of energy conservation in a scattering process. It states, in effect,

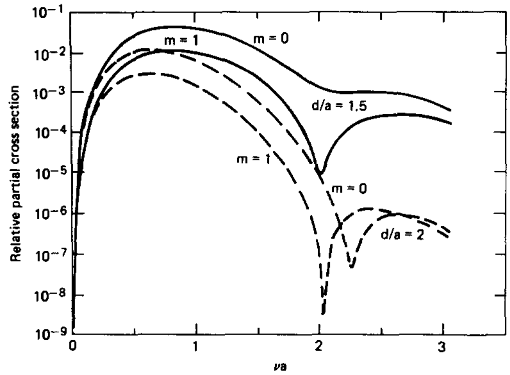

Figure 4a. Exact normalized partial cross sections *vs.* va for $m = 0, 1$.

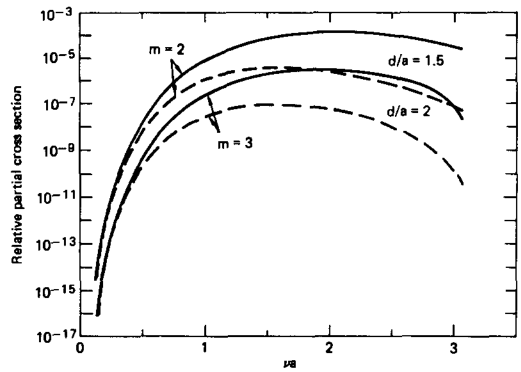

Figure 4b. Exact normalized partial cross sections *vs.* va for $m = 2, 3$.

that in such a process the energy flux of the incident wave is reduced by precisely that amount of energy which appears in the scattered wave when integrated over all scattering angles.

The form of this theorem applicable to the scattering of a surface wave is most easily derived by using Eqs. (24)–(26) to express the energy flux as proportional to the quantity

$$S = i(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{37}$$

By use of Green's theorem and the Laplace equation, Eq. (3a), satisfied by ψ, the power P_s^v leaving any volume v through its surface s can be shown to vanish:

$$\begin{aligned} P_s^v &\propto \oint_s \boldsymbol{S}\cdot \boldsymbol{i}_n ds = i\oint_s (\psi^*\nabla\psi - \psi\nabla\psi^*)\cdot \boldsymbol{n} ds \\ &= i\int_v (\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)dv = 0, \end{aligned} \tag{38}$$

where $\boldsymbol{n}$ is the unit outward normal to s. If one expresses ψ as a sum of incident and scattered velocity potentials, $\psi = \psi_i + \psi_s$, this integral breaks up into four terms:

$$F_{ii} + F_{is} + F_{si} + F_{ss} = 0, \tag{39}$$

with the F's defined as

$$F_{pq} = i \oint_s (\psi_p^* \nabla \psi_q - \psi_p \nabla \psi_q^*) \cdot \boldsymbol{n}\, ds, \tag{40}$$

where p and q can each be either "i" or "s".

It is convenient to take v to be a vertical circular cylinder of radius $\rho \gg a$, with its axis passing through the origin, and extending from the fluid surface to infinite depth. The integrals over the top and bottom surfaces of this cylinder clearly vanish. By using Eq. (5) for ψ_i and the asymptotic form, Eq. (19), for ψ_s, the lateral integral is evaluated in Appendix D, where it is shown that

$$F_{ii} = 0, \tag{41a}$$

$$F_{is} + F_{si} \sim -\frac{2(2\pi)^{\frac{1}{2}}}{v} e^{-2vd} \sum_{m=0}^{\infty} \varepsilon_m \operatorname{Re}(A_m e^{i\pi/4}), \tag{41b}$$

and

$$F_{ss} \sim -\frac{2\pi}{v} e^{-4vd} \sum_{m=0}^{\infty} \varepsilon_m |A_m|^2. \tag{41c}$$

Substitution into Eq. (39) then results in the Optical Theorem appropriate to surface-wave scattering,

$$\sum_{m=0}^{\infty} \varepsilon_m |A_m|^2 = -(2/\pi)^{\frac{1}{2}} e^{2vd} \sum_{m=0}^{\infty} \varepsilon_m \operatorname{Re}(A_m e^{i\pi/4}), \tag{42}$$

or equivalently, in terms of the scattering cross section, σ,

$$\sigma = -\frac{(8\pi)^{\frac{1}{2}}}{v} e^{-2vd} \sum_{m=0}^{\infty} \varepsilon_m \operatorname{Re}(A_m e^{i\pi/4}). \tag{43}$$

To determine how many expansion coefficients, A_m, are needed for an accurate cross section calculation, one may use Eq. (42) as a test. One must employ enough coefficients so that the finite-sum approximations to the two sides of that equation differ by less than the amount prescribed by the desired accuracy.

6. Discussion

One of the most striking characteristics of the cross section for the scattering of a surface wave by a submerged sphere is its essentially single-peak shape as a function of va, illustrated in Figs. 2 and 3. This is in sharp contrast to the well-known oscillatory structure of the cross section for Mie scattering of electromagnetic or scalar waves by a sphere [13].

The reason that these oscillations are largely washed out in the case of surface wave scattering lies in the exponential decay of these waves with depth. If this decay were omitted from the calculation of Ψ_s^B, for example — i.e., if the factor $\exp(va\cos\theta')$ were absent from the integrand of Eq. (B-12) — then the hypothetical scattering velocity potential, $(\Psi_s^B)_{\text{hyp}}$, that such an omission would lead to is given in the Born approximation, asymptotically, by

$$[\Psi_s^B(r)]_{\text{hyp}} \sim -(8\pi/v\rho)^{\frac{1}{2}}(va)^2 \sin\frac{\varphi}{2}\, e^{i\pi/4} e^{iv\rho} j_1\left(2va\sin\frac{\varphi}{2}\right), \tag{44}$$

where j_1 is the spherical Bessel function of order 1, expressible in terms of elementary functions by

$$j_1(z) = (\pi/2z)^{\frac{1}{2}} J_{\frac{3}{2}}(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}. \tag{45}$$

The details are provided in Appendix E, where the corresponding normalized scattering cross section is also shown to be

$$\frac{\sigma_B^{\text{hyp}}}{2a} = \pi^2\{va[J_0(4va) + 1] - J_1(4va)\}. \tag{46}$$

Thus we see that both for the differential scattering cross section and for the total scattering cross section for this hypothetical case, the *nonoscillatory*, modified Bessel functions i_1, I_0, and I_1 of Eqs. (32) and (35) are replaced by the *oscillatory*, ordinary Bessel functions j_1, J_0, and J_1. Thus it is indeed the depth dependence, e^{vz}, of the incident velocity potential, Ψ_i, which largely eliminates the oscillations in the dependence of the scattering cross section on va.

This same depth dependence can also explain the factor e^{-4vd} in the formula for the cross section, Eq. (27). The influence of the surface motion on the submerged sphere is attenuated by a factor e^{-vd}. The resulting modification of the surface due to interaction of the sphere with this attenuated wave is further attenuated by another factor e^{-vd}. Thus Ψ_s should be proportional to e^{-2vd}, and the cross section, which involves the integral of $|\Psi_s|^2$, to e^{-4vd}.

It is interesting to estimate the position of the peak in the curve of the Born approximation scattering cross section, σ_B, as a function of va. This is accomplished by estimating the value of x which maximizes the function

$$f(x) = e^{-\gamma x}[xI_0(x) - 4I_1(x) + x], \tag{47}$$

where $x = 4va$ and $\gamma = d/a$. A power series expansion of the quantity in brackets yields

$$f(x) = \frac{x^5 e^{-\gamma x}}{16} \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n+2)(n+3)!\,n!}. \tag{48}$$

If the second term of this series is small compared to the first, i.e., for $x \ll 5$ (or $va \ll 1.25$), this function has its maximum when $x = 5/\gamma$. In other words, so long as $d/a \gg 1$, the maximum of the Born approximation occurs approximately at $va = 1.25a/d$. For one of the cases shown in Fig. 2, namely $d/a = 2$, this approximation places the maximum at $\nu a = 0.625$, very close to its actual value of 0.68. If $d/a > 2$, the agreement will be even closer. Thus this approximation for the location of the peak of the Born approximation is excellent whenever the Born approximation itself is valid.

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Appendix A. Expansion in spherical harmonics

We shall expand all the terms of the Green's function, Eq. (7), as well as the incident surface wave ψ_i, Eq. (5), in spherical harmonics with respect to the center of the spherical obstacle.

i) The expansion of $1/|\boldsymbol{r} - \boldsymbol{r}'|$ is well known [14],

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \sum_{m,n} \varepsilon_m \frac{(n-m)!}{(n+m)!} \frac{(r'_<)^n}{(r'_>)^{n+1}} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\varphi - \varphi'), \tag{A-1}$$

where (θ, φ) and (θ', φ') are the angular spherical coordinates of $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively,

$$\varepsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{otherwise,} \end{cases}$$

$r'_< = \min(r, r')$, $r_> = \max(r, r')$, and the summation $\sum_{m,n}$ denotes $\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}$.

ii) Since $\psi_i = e^{\nu(z-d)+i\nu x}$ has the form of a plane wave, but with its component of the wave vector in the z-direction pure imaginary, we shall obtain its expansion from the well known spherical harmonic expansion of the plane wave [15],

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{m,n} \varepsilon_m (2n+1) i^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\alpha) \cos m(\varphi - \beta) j_n(kr) \tag{A-2}$$

where $\boldsymbol{k} = (k, \alpha, \beta)$ in spherical coordinates, and $j_n(z) = (\pi/2z)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z)$ is the spherical Bessel function of order n.

In order that $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ be proportional to ψ_i, the cartesian components of $\boldsymbol{k}$ must have the values

$$k_x = k \sin\alpha \cos\beta = \nu, \tag{A-3a}$$

$$k_y = k \sin\alpha \sin\beta = 0, \tag{A-3b}$$

and

$$k_z = k \cos\alpha = -i\nu. \tag{A-3c}$$

Eqs. (A-3a) and (A-3b) require that $\beta = 0$, so that k and α are determined from

$$k \sin \alpha = \nu, \quad k \cos \alpha = -i\nu. \tag{A-4a, b}$$

But these equations are inconsistent for real α, since they require that

$$k^2 \sin^2 \alpha + k^2 \cos^2 \alpha = k^2 = 0,$$

which can only occur if $\sin \alpha$ and $\cos \alpha$ are infinite. We must therefore resort to a limiting procedure.

To that end, we shall express the arguments of P_n^m and j_n in terms of ν and $k_z = k \cos \alpha$. Eventually we shall let $k_z \to -i\nu$. We define $\xi = (1 + \nu^2/k_z^2)^{-\frac{1}{2}}$, so that $k = k_z/\xi$. Then the required limit is $\lim_{\xi \to \infty} P_n^m(\xi) j_n(-i\nu r/\xi)$.

Note that for real α, the associated Legendre polynomial $P_n^m(\cos \alpha)$ in Eq. (A-2), from which the factor $P_n^m(\xi)$ in the above limit arises, has an argument between -1 and $+1$. It is therefore defined in the usual way [15] as

$$P_n^m(\cos \alpha) = (-\sin \alpha)^m \frac{d^m}{d(\cos \alpha)^m} P_n(\cos \alpha). \tag{A-5a}$$

Thus $P_n^m(\xi)$ must be defined as

$$P_n^m(\xi) = (-1)^m (1 - \xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi), \tag{A-5b}$$

which differs by a factor i^m from the usual definition for associated Legendre polynomials with an argument greater than 1 [16]. This factor must be inserted in the tabulated asymptotic limit [17]. Thus we have for large ξ,

$$P_n^m(\xi) \sim \frac{i^m (2\xi)^n \Gamma(n + \frac{1}{2})}{\pi^{\frac{1}{2}} (n - m)!}. \tag{A-6}$$

Using the first term in the series expansion of $j_n(z)$ [18],

$$j_n(z) \sim \frac{z^n}{(2n + 1)!!}, \tag{A-7}$$

where $(2n + 1)!! = (2n + 1)(2n - 1) \ldots 3 \times 1$, we obtain for the limit

$$\lim_{\xi \to \infty} P_n^m(\xi) j_n\left(-\frac{i\nu r}{\xi}\right) = \frac{i^{m-n} (\nu r)^n}{(2n + 1)(n - m)!}, \tag{A-8}$$

which leads to

$$\psi_i = e^{\nu(z-d) + i\nu x} = e^{-\nu d} \sum_{m,n} \frac{\varepsilon_m i^m (\nu r)^n}{(m + n)!} \cos m\varphi P_n^m(\cos \theta). \tag{A-9}$$

iii) For an expansion of $1/|\boldsymbol{r}-\boldsymbol{r}''|$ we start with a formula like Eq. (A-1),

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}=\sum_{m,n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\frac{(r''_<)^n}{(r''_>)^{n+1}}P_n^m(\cos\theta)P_n^m(\cos\theta'')\cos m(\varphi-\varphi'), \tag{A-10}$$

where $r''_< = \min(r, r'')$, $r''_> = \max(r, r'')$, and $(r'', \theta'', \varphi')$ are the spherical coordinates of $\boldsymbol{r}''$ (see Fig. 1). Since both P and P' will eventually lie on the surface of the submerged sphere when evaluating $(\partial^2 G/\partial r\partial r')_{r=r'=a}$ in Eq. (9), we have $r''_< = r$ and $r''_> = r''$. To facilitate the θ'-integration in Eq. (9), we will now express $P_n^m(\cos\theta'')/(r'')^{n+1}$ in terms of θ' and r'.

To that end, we note that this quantity is also equal to $P_n^m[\cos(\pi-\theta_1')]/(r_1')^{n+1}$ (where $\boldsymbol{r}_1'$ is the vector from 0_1, the reflection of 0, to P'), which is the irregular solid spherical harmonic with respect to a center displaced by $2d$ from 0. The transformation of such harmonics has been accomplished by Steinborn and Ruedenberg [19]. They find that, for $r' < 2d$,

$$P_n^m[\cos(\pi-\theta_1')]=\sum_{\kappa=m}^{\infty}\frac{(n+\kappa)!}{(n-m)!\,(\kappa+m)!}\left(\frac{r_1}{2d}\right)^{n+1}\left(\frac{r'}{2d}\right)^{\kappa}P_\kappa^m(\cos\theta'), \tag{A-11}$$

which leads to the required result,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}=\frac{1}{2d}\sum_{\kappa,m,n}\varepsilon_m\frac{(n+\kappa)!}{(n+m)!\,(\kappa+m)!}\left(\frac{r}{2d}\right)^{n}\left(\frac{r'}{2d}\right)^{\kappa}P_n^m(\cos\theta)P_\kappa^m(\cos\theta')$$
$$\times\cos m(\varphi-\varphi') \tag{A-12}$$

for the desired case for which $r' < d$, where the triple summation $\sum_{\kappa,m,n}$ denotes $\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{\kappa=m}^{\infty}$.

iv) To obtain the expansion coefficients $A_{mn\kappa}$ in the spherical harmonic expansion of $2\pi i\nu e^{\nu(z+z'-2d)}J_0(\nu R)$,

$$2\pi i\nu e^{\nu(z+z'-2d)}J_0(\nu R)=\sum_{\kappa,m,n}\varepsilon_m A_{mn\kappa}P_n^m(\cos\theta)P_\kappa^m(\cos\theta')\cos m(\varphi-\varphi'), \tag{A-13}$$

where $R = [(x-x')^2+(y-y')^2]^{\frac{1}{2}}$, we express z and z' in spherical polar coordinates, substitute for $J_0(\nu R)$ the addition theorem expansion,

$$J_0(\nu R)=\sum_{m=0}^{\infty}\varepsilon_m J_m(\nu r\sin\theta)J_m(\nu r'\sin\theta')\cos m(\varphi-\varphi'), \tag{A-14}$$

multiply the resulting equation by

$$P_l^p(\cos\theta)P_j^q(\cos\theta')\sin\theta\sin\theta'\cos p\varphi\cos q\varphi',$$

and integrate the expression over φ and φ' from 0 to 2π and over θ and θ' from 0 to π. Remembering the orthogonality of the trigonometric functions and of the associated Legendre polynomials, as well as their normalization integrals,

$$\int_0^\pi [P_l^m(\cos\theta)]^2 \sin\theta\, d\theta = \left(\frac{2}{2l+1}\right)\frac{(l+m)!}{(l-m)!}, \tag{A-15}$$

we obtain for the expansion coefficients

$$A_{mn\kappa} = \tfrac{1}{2}\pi i\nu e^{-2\nu d}\,\frac{(2n+1)(2\kappa+1)(n-m)!\,(\kappa-m)!}{(n+m)!\,(\kappa+m)!}$$

$$\times \int_0^\pi e^{\nu r\cos\theta} J_m(\nu r\sin\theta) P_n^m(\cos\theta)\sin\theta\, d\theta$$

$$\times \int_0^\pi e^{\nu r'\cos\theta'} J_m(\nu r'\sin\theta') P_\kappa^m(\cos\theta')\sin\theta'\, d\theta'. \tag{A-16}$$

The remaining integrals can be evaluated from a form of Gegenbauer's finite integral, namely [20]

$$\int_0^\pi e^{ikr\cos\theta\cos\alpha} J_m(kr\sin\theta\sin\alpha) P_n^m(\cos\theta)\sin\theta\, d\theta = 2i^{n-m} P_n^m(\cos\alpha) j_n(kr). \tag{A-17}$$

To obtain the integrals in Eq. (A-16) from this equation requires setting

$$k\sin\alpha = \nu, \quad k\cos\alpha = -i\nu, \tag{A-18a, b}$$

the same as in Eqs. (A-4) The same limiting procedure must therefore be used. The result is

$$\int_0^\pi e^{\nu r\cos\theta} J_m(\nu r\sin\theta) P_n^m(\cos\theta)\sin\theta\, d\theta = \frac{2(\nu r)^n}{(n-m)!\,(2n+1)}, \tag{A-19}$$

which leads to the value for the expansion coefficients

$$A_{mn\kappa} = \frac{2\pi i\nu e^{-2\nu d}(\nu r)^n(\nu r')^\kappa}{(m+n)!\,(m+\kappa)!}. \tag{A-20}$$

v) To obtain the expansion coefficients $B_{mn\kappa}$ in the spherical harmonic expansion

$$2\nu \fint_0^\infty e^{k(z+z'-2d)}\frac{J_0(kR)}{k-\nu}\, dk$$

$$= \sum_{m,n,\kappa} \varepsilon_m B_{mn\kappa} P_n^m(\cos\theta) P_\kappa^m(\cos\theta')\cos m(\varphi-\varphi'), \tag{A-21}$$

we multiply by $P_l^p(\cos\theta)P_j^q(\cos\theta')\sin\theta\sin\theta'\cos p\varphi\cos q\varphi'$ and integrate over φ and φ' from 0 to 2π, and over θ and θ' from 0 to π, with the result

$$B_{mn\kappa} = \frac{2\nu(-1)^m r^n r'^\kappa}{(m+n)!\,(m+\kappa)!}\fint_0^\infty \frac{e^{-2kd}}{k-\nu} k^{n+\kappa}\, dk. \tag{A-22}$$

The integral in Eq. (A-22) can be expressed in terms of derivatives of the exponential integral,

$$\int_0^\infty \frac{e^{-2kd}}{k-\nu} k^{n+\kappa} dk = (-\nu)^{n+\kappa+1} \frac{\partial^{n+\kappa}}{\partial p^{n+\kappa}} [e^{-p}\overline{Ei}(p)]_{p=2\nu d}, \tag{A-23}$$

with $\overline{Ei}(p)$ defined as

$$\overline{Ei}(p) = -\int_{-p}^\infty \frac{e^{-t}}{t} dt \text{ for } p > 0. \tag{A-24}$$

The derivatives can be evaluated as

$$\frac{\partial^l}{\partial p^l} [e^{-p}\overline{Ei}(p)] = (-1)^l \left[e^{-p}\overline{Ei}(p) - \sum_{j=0}^{l-1} \frac{j!}{p^{j+1}} \right]. \tag{A-25}$$

Thus the final result is

$$B_{mn\kappa} = \frac{2\nu(-1)^{m+1}(\nu r)^n(\nu r')^\kappa}{(m+n)!\,(m+\kappa)!} \left[e^{-2\nu d}\overline{Ei}(2\nu d) - \sum_{j=0}^{n+\kappa-1} \frac{j!}{(2\nu d)^{j+1}} \right]. \tag{A-26}$$

Appendix B. Asymptotic expansion of scattered velocity potential and of its Born approximation

The scattered velocity potential, $\psi_s = \psi - \psi_i$ is written in terms of the Green's function G and the total velocity potential ψ, from Eq. (4),

$$\psi_s(\boldsymbol{r}) = 1/4\pi \int_s \psi(\boldsymbol{r}')\nabla' G(\boldsymbol{r}, \boldsymbol{r}')\cdot \boldsymbol{n}' d^2 s'. \tag{B-1}$$

To obtain the asymptotic form for ψ_s, therefore, requires the asymptotic form of $G(\boldsymbol{r}, \boldsymbol{r}')$ for $r \to \infty$ and $r' = a$. Thus

$$\psi_s(\boldsymbol{r}) \sim \frac{a^2}{4\pi} \int_0^{2\pi} d\varphi' \int_0^\pi \psi(a, \theta', \varphi') \left[\frac{\partial}{\partial r'} G(\boldsymbol{r}, \boldsymbol{r}') \right]_{r'=a,\, r\to\infty} \sin\theta' d\theta'. \tag{B-2}$$

To obtain the asymptotic limit of G, we rewrite it as [11]

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|} - \frac{4\nu}{\pi} \times \int_0^\infty \{\nu \cos[k(2d-z-z')] + k \sin[k(2d-z-z')]\} \times \frac{K_0(kR)}{k^2+\nu^2} dk + 2\pi i \nu e^{-\nu(2d-z-z')} H_0^{(1)}(\nu R), \tag{B-3}$$

where K_0 is the modified Bessel function of the second kind of order zero and $H_0^{(1)}$ is the Hankel function of the first kind of order zero. The leading term in the asymptotic expansion of this expression for large r but finite r' comes from the last term, so that

$$G(\mathbf{r}, \mathbf{r}') \sim 2\pi i \nu \mathrm{e}^{-\nu(2d-z-z')} \mathrm{e}^{i(\nu R - \frac{1}{4}\pi)} (2/\pi \nu R)^{\frac{1}{2}} [1 + 0(1/R)], \tag{B-4}$$

as terms of order $1/R$ from the first three terms of Eq. (B-3) cancel.

In order to evaluate $\partial G/\partial r'$, Eq. (B-4) must first be expressed in spherical coordinates of the "primed" variables. The leading terms for large r in $\partial G/\partial r'$ arise from differentiating the exponentials, so that

$$\left[\frac{\partial G}{\partial r'}(\mathbf{r}, \mathbf{r}')\right]_{r'=a} \sim 2\pi i \nu^2 (2/\pi \nu \rho)^{\frac{1}{2}} \mathrm{e}^{-\nu(2d-z-a\cos\theta')} \mathrm{e}^{i\nu[r\sin\theta - a\sin\theta'\cos(\varphi-\varphi')]} \mathrm{e}^{-i\pi/4}$$
$$\times [\cos\theta' - i\sin\theta\sin\theta'\cos(\varphi-\varphi')]. \tag{B-5}$$

Replacing ψ in Eq. (B-2) by its spherical harmonic form of Eq. (17), enables us to write the asymptotic limit for ψ_s as

$$\Psi_s(\mathbf{r}) \sim \mathrm{e}^{i\pi/4} (\nu a)^2 \mathrm{e}^{-\nu(2d-z)} (2\pi \nu d)^{-\frac{1}{2}} \mathrm{e}^{i\nu\rho} \sum_{m,n} \varepsilon_m f_{mn} \left(\frac{2n+1}{n+1}\right) \frac{\partial}{\partial(\nu a)}$$
$$\times \int_0^{2\pi} d\varphi' \cos m\varphi' \int_0^{\pi} d\theta' \sin\theta' P_n^m(\cos\theta') \mathrm{e}^{\nu a[\cos\theta' - i\sin\theta'\cos(\varphi-\varphi')]} \tag{B-6}$$

since for $r \to \infty$ but z finite — the desired asymptotic regime — $\theta \to \pi/2$.

The φ'-integral is easily evaluated,

$$\int_0^{2\pi} \mathrm{e}^{-i\nu a\sin\theta'\cos(\varphi'-\varphi)} \cos m\varphi' d\varphi' = 2\pi i^{-m} J_m(\nu a \sin\theta') \cos m\varphi, \tag{B-7}$$

leaving us with

$$\psi_s(\mathbf{r}) \sim \pi \mathrm{e}^{i\pi/4} (\nu a)^2 \mathrm{e}^{-\nu(2d-z)} (2/\pi\nu\rho)^{\frac{1}{2}} \sum_{m,n} \varepsilon_m f_{mn} i^m \left(\frac{2n+1}{n+1}\right) \cos m\varphi \frac{\partial}{\partial(\nu a)}$$
$$\times \int_0^{\pi} d\theta' \sin\theta' P_n^m(\cos\theta') J_m(\nu a \sin\theta') \mathrm{e}^{\nu a\cos\theta'}. \tag{B-8}$$

But this contains the same integral as has been evaluated in Eq. (A-19), so that we obtain finally for the desired asymptotic limit of the scattered velocity potential

$$\psi_s(\mathbf{r}) \sim (8\pi/\nu\rho)^{\frac{1}{2}} \mathrm{e}^{-\nu(2d-z)} \mathrm{e}^{i\nu\rho} \mathrm{e}^{i\pi/4} \sum_{m,n} \varepsilon_m f_{mn} \frac{i^{-m} n(\nu a)^{n+1}}{(n+1)(n-m)!} \cos m\varphi. \tag{B-9}$$

The Born approximation, ψ_s^B, is defined by an equation similar to Eq. (B-1), but with $\psi(\mathbf{r}')$ replaced by $\psi_i(\mathbf{r}')$, i.e.,

$$\psi_s^B(\mathbf{r}) = 1/4\pi \int_s \psi_i(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{n}' d^2 s'. \tag{B-10}$$

Using the spherical harmonic expansion of ψ_i, Eqs. (10e) and (11e),

$$\psi_i(r') = e^{-\nu d} \sum_{m,n} \varepsilon_m \frac{i^m(\nu r')^n}{(m+n)!} P_n^m(\cos\theta')\cos m\varphi', \tag{B-11}$$

and the asymptotic form of Eq. (B-5) for $[\partial G(r, r')/\partial r']_{r'=a}$, we have for the asymptotic form of ψ_s^B an expression similar to Eq. (B-6),

$$\psi_s^B(r) \sim e^{i\pi/4}(\nu a)^2 e^{-\nu(3d-z)}(2\pi\nu d)^{-\frac{1}{2}} e^{i\nu\rho} \sum_{m,n} \varepsilon_m \frac{i^m(\nu a)^n}{(m+n)!}\frac{\partial}{\partial(\nu a)}$$
$$\times \int_0^{2\pi} d\varphi' \cos m\varphi' \int_0^{\pi} d\theta' \sin\theta' P_n^m(\cos\theta') e^{\nu a[\cos\theta' - i\sin\theta'\cos(\varphi-\varphi')]}. \tag{B-12}$$

The integrals have already been evaluated in Eqs. (B-7) and (A-19), so that we obtain

$$\psi_s^B \sim e^{i\pi/4} e^{-\nu(3d-z)}(8\pi/\nu\rho)^{\frac{1}{2}} e^{i\nu\rho} \sum_{m,n} \varepsilon_m \cos m\varphi \frac{n(\nu a)^{2n+1}}{(2n+1)(n-m)!\,(n+m)!}. \tag{B-13}$$

The sums can be evaluated in closed form. We sum first over n,

$$\sum_{n=m}^{\infty} \frac{n(\nu a)^{2n+1}}{(2n+1)(n-m)!\,(n+m)!}.$$

We define

$$R_m(x) = \sum_{n=m}^{\infty} \frac{n x^{2n+1}}{(2n+1)(n-m)!\,(n+m)!}. \tag{B-14}$$

Differentiating, dividing by x, and integrating the result from 0 to x gives, for $m > 0$,

$$\int_0^x R_m'(t)\frac{dt}{t} = \tfrac{1}{2} \sum_{n=m}^{\infty} \frac{x^{2n}}{(n-m)!\,(n+m)!}. \tag{B-15}$$

For $m = 0$, a similar procedure yields

$$\int_0^x R_0'(t)\frac{dt}{t} = \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{x^{2n}}{(n!)^2}. \tag{B-16}$$

The sums in Eqs. (B-15) and (B-16) can be expressed in terms of the modified Bessel function $I_p(z)$, which has a series expansion

$$I_p(z) = \sum_{l=0}^{\infty} \frac{(z/2)^{2l+p}}{l!\,\Gamma(l+p+1)}. \tag{B-17}$$

Thus Eq. (B-16) becomes directly

$$\int_0^x R_0'(t)\frac{dt}{t} = \tfrac{1}{2}[I_0(2x) - 1]. \tag{B-18}$$

To evaluate the sum in Eq. (B-15), we set $l = n - m$, with the result

$$\int_0^x R_m'(t)\frac{dt}{t} = \tfrac{1}{2}I_{2m}(2x) \text{ for } m > 0. \tag{B-19}$$

Differentiating Eqs. (B-18) and (B-19) yields

$$R_m'(x) = xI_{2m}'(2x) \text{ for all } m. \tag{B-20}$$

$R_m(x)$ can be expressed in terms of $I_{2m}(x)$ by means of an integration by parts,

$$R_m(x) = \frac{x}{2}I_{2m}(2x) - \tfrac{1}{4}\int_0^{2x} I_{2m}(t)dt. \tag{B-21}$$

Thus the asymptotic form for ψ_s^B reduces to

$$\psi_s^B(\boldsymbol{r}) \sim e^{i\pi/4}e^{-\nu(3d-z)}e^{i\nu\rho}(2\pi/\nu\rho)^{\frac{1}{2}}$$
$$\times\left[\nu a \sum_{m=0}^{\infty}\varepsilon_m I_{2m}(2\nu a)\cos m\varphi - \tfrac{1}{2}\int_0^{2\nu a}\sum_{m=0}^{\infty}\varepsilon_m I_{2m}(t)\cos m\varphi\, dt\right]. \tag{B-22}$$

The sum over m can also be evaluated in closed form [21],

$$\sum_{m=0}^{\infty}\varepsilon_m I_{2m}(p)\cos m\varphi = \cosh\left(p\cos\frac{\varphi}{2}\right). \tag{B-23}$$

The integral can now be evaluated trivially, so that we have, finally,

$$\psi_s^B(\boldsymbol{r}) \sim e^{i\pi/4}e^{-\nu(3d-z)}e^{i\nu\rho}(2\pi/\nu\rho)^{\frac{1}{2}}$$
$$\times\left[\nu a\cosh\left(2\nu a\cos\frac{\varphi}{2}\right) - \frac{\sinh\left(2\nu a\cos\frac{\varphi}{2}\right)}{2\cos\frac{\varphi}{2}}\right], \tag{B-24}$$

or, in terms of the modified spherical Bessel function defined in Eq. (33)

$$\psi_s^B(\boldsymbol{r}) \sim e^{i\pi/4}e^{-\nu(3d-z)}(8\pi/\nu\rho)^{\frac{1}{2}}(\nu a)^2 e^{i\nu\rho}\cos\frac{\varphi}{2}\, i_1\left(2\nu a\cos\frac{\varphi}{2}\right). \tag{B-25}$$

Appendix C. Born approximation to the scattering cross section

We wish to calculate the Born approximation to the scattering cross section

$$\sigma_B = 2\nu\rho\int_{-\infty}^{d} dz\int_0^{2\pi}|(\psi_s^B)_{\text{asymp}}|^2 d\varphi, \tag{C-1}$$

where $(\psi_s^B)_{\text{asymp}}$ is the asymptotic form of the Born approximation to the scattered velocity potential given in Eqs. (B-24) and (B-25). The z-integral is trivial. The φ-integral requires the evaluation of

$$K(\gamma) = \int_0^{2\pi} \left[\cosh\left(\frac{\gamma}{2}\cos\frac{\varphi}{2}\right) - \frac{\sinh\left(\frac{\gamma}{2}\cos\frac{\varphi}{2}\right)}{\frac{\gamma}{2}\cos\frac{\varphi}{2}} \right]^2 d\varphi \tag{C-2}$$

with $\gamma = 4va$. Changing the integration variable to $t = \cos\frac{1}{2}\varphi$ and using the double angle formulas for hyperbolic functions reduces K to

$$K(\gamma) = 2\int_0^1 \frac{\cosh \gamma t}{(1-\gamma^2)^{\frac{1}{2}}}\, dt + \\ + \frac{8}{\gamma^2}\int_0^1 \frac{(\cosh \gamma t - 1)}{t^2(1-t^2)^{\frac{1}{2}}}\, dt - \frac{8}{\gamma}\int_0^1 \frac{\sinh \gamma t}{t(1-t^2)^{\frac{1}{2}}}\, dt + \pi. \tag{C-3}$$

The integrals can all be evaluated with the aid of the well-known integral representation for $I_0(\gamma)$ [22],

$$I_0(\gamma) = \frac{2}{\pi}\int_0^1 \frac{\cosh \gamma t}{(1-\gamma^2)^{\frac{1}{2}}}\, dt. \tag{C-4}$$

For example, if we let

$$\mathscr{T}(\gamma) = \int_0^1 \frac{\sinh \gamma t}{t(1-t^2)^{\frac{1}{2}}}\, dt, \tag{C-5}$$

then

$$\mathscr{T}'(\gamma) = \frac{\pi}{2} I_0(\gamma), \tag{C-6}$$

and

$$\mathscr{T}(\gamma) = \frac{\pi}{2}\int_0^\gamma I_0(t)\, dt. \tag{C-7}$$

Similarly, if we set

$$\mathscr{L}(\gamma) = \int_0^1 \frac{(\cosh \gamma t - 1)}{t^2(1-t^2)^{\frac{1}{2}}}\, dt, \tag{C-8}$$

then

$$\mathscr{L}'(\gamma) = \mathscr{T}(\gamma) = \frac{\pi}{2}\int_0^\gamma I_0(t)\, dt, \tag{C-9}$$

and

$$\mathscr{L}(\gamma) = \frac{\pi}{2}\int_0^\gamma dv \int_0^v I_0(t)\, dt. \tag{C-10}$$

Changing the order of integration results in

$$\mathscr{L}(\gamma) = \frac{\pi}{2}\int_0^\gamma I_0(t)dt \int_t^\gamma dv = \frac{\pi}{2}\int_0^\gamma (\gamma - t)I_0(t)dt. \tag{C-11}$$

The second integral can be evaluated in closed form,

$$\int_0^\gamma t I_0(t)dt = \gamma I_1(\gamma), \tag{C-12}$$

so that the expression for $\mathscr{L}$ becomes

$$\mathscr{L}(\gamma) = \frac{\pi\gamma}{2}\left[\int_0^\gamma I_0(t)dt - I_1(\gamma)\right]. \tag{C-13}$$

Substituting Eqs. (C-7) and (C-13) in (C-3), we note that the terms involving $\int_0^\gamma I_0(t)dt$ cancel, and that

$$K(\gamma) = \pi\left[1 + I_0(\gamma) - \frac{4}{\gamma} I_1(\gamma)\right]. \tag{C-14}$$

Substituting in Eq. (C-1) gives, finally, for the Born approximation to the normalized cross section

$$\frac{\sigma_B}{2a} = \pi^2 e^{-4\nu d}\{\nu a[I_0(4\nu a) + 1] - I_1(4\nu a)\}. \tag{C-15}$$

Appendix D. Calculation of the *F*'s

In evaluating F_{ii}, F_{is}, F_{si} and F_{ss}, defined by Eq. (40), we use for ψ_i the plane wave of Eq. (5), as well as its expansion in polar coordinates,

$$\psi_i = e^{\nu(z-d)+i\nu x} = e^{\nu(z-d)} \sum_{m=-\infty}^{\infty} i^m e^{im\varphi} J_m(\nu\rho), \tag{D-1}$$

and for ψ_s, the asymptotic form of Eq. (B-9),

$$\psi_s \sim \frac{e^{i\nu\rho}}{(\nu\rho)^{\frac{1}{2}}} e^{\nu(z-3d)} \sum_{m=-\infty}^{\infty} A_m e^{im\varphi}, \tag{D-2}$$

with

$$A_m = (8\pi)^{\frac{1}{2}} i^{-|m|} e^{i\pi/4} \sum_{n=|m|}^{\infty} \frac{(\nu a)^{n+1} n\bar{f}_{|m|n}}{(n+1)(n-|m|)!}, \tag{D-3}$$

and $\bar{f}_{mn} = e^{\nu d} f_{mn}$. The integral over the lateral surface can be written as

$$\oint_s \ldots ds = \rho \int_{-\infty}^{d} \ldots dz \int_0^{2\pi} \ldots d\varphi. \tag{D-4}$$

The z-dependence of all terms comprising the F's is $e^{2\nu z}$, so that the integral over z yields $e^{2\nu d}/2\nu$. The quantity F_{ii} is easily shown to vanish, using

$$\nabla\psi_i \cdot \boldsymbol{i}_n = \frac{\partial\psi_i}{\partial x} \boldsymbol{i}_x \cdot \boldsymbol{i}_\rho = i\nu \cos\varphi e^{\nu(z-d)+i\nu x}, \tag{D-5}$$

so that

$$F_{ii} = -\rho \int_0^{2\pi} \cos\varphi d\varphi = 0. \tag{D-6}$$

For F_{is} we have by straight-forward substitution, and carrying out the integration over φ,

$$F_{is} \sim -\frac{1}{\nu}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-2\nu d} \sum_{m=-\infty}^{\infty} \{A_m[(-1)^m e^{-i\pi/4} e^{2i\nu\rho} + e^{i\pi/4}] + \text{c.c.}\}, \tag{D-7}$$

where c.c. denotes the complex conjugate. Similarly,

$$F_{si} \sim \frac{1}{\nu}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-2\nu d} \sum_{m=-\infty}^{\infty} \{A_m[(-1)^m e^{-i\pi/4} e^{2i\nu\rho} - e^{i\pi/4}] + \text{c.c.}\}. \tag{D-8}$$

In the sum $F_{is} + F_{si}$, therefore, the ρ-dependent terms cancel, and we are left with

$$F_{is} + F_{si} \sim -\frac{(2\pi)^{\frac{1}{2}}}{\nu} e^{-2\nu d} \sum_{m=-\infty}^{\infty} (A_m e^{i\pi/4} + \text{c.c.})$$

$$= -\frac{2(2\pi)^{\frac{1}{2}}}{\nu} e^{-2\nu d} \sum_{m=0}^{\infty} \varepsilon_m \operatorname{Re}(A_m e^{i\pi/4}). \tag{D-9}$$

The calculation of F_{ss} proceeds in the same way, with the result

$$F_{ss} \sim -\frac{2\pi}{\nu} e^{-4\nu d} \sum_{m=0}^{\infty} \varepsilon_m |A_m|^2. \tag{D-10}$$

Appendix E. Born approximation without attenuation with depth

We wish to calculate the Born approximation to the hypothetical velocity potential and the associated scatterings cross section of a submerged sphere in the absence of the attenuating factor $e^{\nu z}$ for the surface wave, i.e.,

$$[\Psi_s^B(\boldsymbol{r})]_{\text{hyp}} = 1/4\pi \int_S \Psi_i(\boldsymbol{r}')\nabla' G_{\text{hyp}}(\boldsymbol{r}, \boldsymbol{r}')\cdot \boldsymbol{n}' d^2 s \tag{E-1}$$

where $\Psi_i(\boldsymbol{r}') = e^{i\nu x'} = e^{i\nu a \sin\theta' \cos\varphi'}$ and

$$G_{\text{hyp}}(\boldsymbol{r}, \boldsymbol{r}') \sim 2\pi i\nu H_0^{(1)}(\nu R) \sim 2\pi i\nu (2/\pi\nu R)^{\frac{1}{2}} e^{i(\nu R - \pi/4)}. \tag{E-2}$$

Expanding R for $\rho \gg a$ and differentiating, yields

$$\left[\frac{\partial G_{\mathrm{hyp}}(\boldsymbol{r}, \boldsymbol{r}')}{\partial r'}\right]_{r'=a} \sim 2\pi v^2 (2/\pi v\rho)^{\frac{1}{2}} \mathrm{e}^{-i\pi/4} \mathrm{e}^{iv\rho} \sin\theta \sin\theta' \cos(\varphi - \varphi') \times \mathrm{e}^{-iva\sin\theta'\cos(\varphi-\varphi')} \tag{E-3}$$

so that in the asymptotic regime where $\theta \to \pi/2$ we have

$$\begin{aligned}[\Psi_s^B(\boldsymbol{r})]_{\mathrm{hyp}} &\sim (2\pi v\rho)^{-\frac{1}{2}} (va)^2 \mathrm{e}^{-i\pi/4} \mathrm{e}^{iv\rho} \int_0^{2\pi} d\varphi' \cos(\varphi - \varphi') \int_0^{\pi} d\theta' \sin^2\theta' \\ &\quad \times \mathrm{e}^{iva\sin\theta'[\cos\varphi' - \cos(\varphi-\varphi')]} \\ &= (2\pi v\rho)^{-\frac{1}{2}} (va)^2 \mathrm{e}^{-i\pi/4} \mathrm{e}^{iv\rho} \int_0^{2\pi} d\varphi' \cos(\varphi - \varphi') \int_0^{\pi} d\theta' \sin^2\theta' \\ &\quad \times \mathrm{e}^{-2iva\sin\theta'\sin\frac{1}{2}\varphi\sin(\varphi' - \frac{1}{2}\varphi)}.\end{aligned} \tag{E-4}$$

The φ'-integral is readily evaluated,

$$\begin{aligned}&\int_0^{2\pi} \mathrm{e}^{-2iva\sin\theta'\sin\frac{1}{2}\varphi\sin(\varphi' - \frac{1}{2}\varphi)} \cos(\varphi' - \varphi) d\varphi' \\ &\quad = -2\pi i \sin\frac{\varphi}{2} J_1\left(2va\sin\theta'\sin\frac{\varphi}{2}\right).\end{aligned} \tag{E-5}$$

For the θ'-integral, we employ Sonine's first integral [23],

$$\int_0^{\pi/2} J_\mu(z\sin\theta') \sin^{\mu+1}\theta' \cos^{2\nu+1}\theta' d\theta' = \frac{2^\nu \Gamma(\nu+1)}{z^{\nu+1}} J_{\mu+\nu+1}(z). \tag{E-6}$$

Setting $\mu = 1$ and $\nu = -\frac{1}{2}$, we obtain

$$[\Psi_s^B(\boldsymbol{r})]_{\mathrm{hyp}} \sim -\pi(2/v\rho)^{\frac{1}{2}} (va)^{\frac{3}{2}} \left(\sin\frac{\varphi}{2}\right)^{\frac{1}{2}} \mathrm{e}^{i\pi/4} \mathrm{e}^{iv\rho} J_{\frac{3}{2}}\left(2va\sin\frac{\varphi}{2}\right), \tag{E-7}$$

or, in terms of the spherical Bessel function,

$$j_1(z) = (\pi/2z)^{\frac{1}{2}} J_{\frac{3}{2}}(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \tag{E-8}$$

$$[\Psi_s^B(\boldsymbol{r})]_{\mathrm{hyp}} \sim -(8\pi/v\rho)^{\frac{1}{2}} (va)^2 \sin\frac{\varphi}{2} \mathrm{e}^{i\pi/4} \mathrm{e}^{iv\rho} j_1\left(2va\sin\frac{\varphi}{2}\right). \tag{E-9}$$

The hypothetical scattering cross section corresponding to this velocity potential is given by

$$\sigma_B^{\mathrm{hyp}} = \rho \int_0^{2\pi} |(\Psi_s^B)_{\mathrm{hyp}}|^2 d\varphi, \tag{E-10}$$

which requires the evaluation of

$$M(\gamma) = \int_0^{2\pi} \left[\cos\left(\frac{\gamma}{2}\sin\frac{\varphi}{2}\right) - \frac{\sin\left(\frac{\gamma}{2}\sin\frac{\varphi}{2}\right)}{\frac{\gamma}{2}\sin\frac{\varphi}{2}} \right]^2 d\varphi. \tag{E-11}$$

Changing variables to $\chi = \varphi - \pi$, and utilizing the periodicity of the integrand to rewrite $\int_{-\pi}^{\pi} d\chi$ as $\int_0^{2\pi} d\chi$, we note that this integral is the same as $K(i\gamma)$ [See Eq. (C-2)]. Thus we have

$$M(\gamma) = \pi\left[1 + J_0(\gamma) - \frac{4}{\gamma} J_1(\gamma)\right] \tag{E-12}$$

and

$$\sigma_B^{\text{hyp}} = 2a\pi^2\{\nu\alpha[J_0(4\nu a) + 1] - J_1(4\nu a)\}. \tag{E-13}$$

REFERENCES

[1] Sir W. Thomson, On stationary waves in flowing water, *Phil. Mag.* (5) 22 (1886) 353–357, 445–452, 517–530 and 23 (1887) 52–58.

[2] Sir Horace Lamb, On some cases of wave-motion in deep water, *Annali di Matematica* (3) 21 (1913) 237–250.

[3] N. E. Kochin, The theory of waves generated by oscillations of a body under the free surface of a heavy, incompressible fluid, *Uchenye Zapiski Moskov. Gos. Univ.*, 46 (1940) 85–106. Transl. in Soc. Nav. Arch. Marine Eng., *Tech. Res. Bull. No. 1-10* (1952).

[4] W. R. Dean, On the reflection of surface waves by a submerged plane barrier, *Proc. Camb. Phil. Soc.* 41 (1945) 231–238.
W. R. Dean, On the reflexion of a surface wave by a submerged cylindrical barrier, *ibid*, 44 (1948) 483–491.

[5] Fritz John, On the motion of floating bodies II. Simple harmonic motion, *Comm. Pure App. Math.*, 3 (1950) 45–101.

[6] Sir Thomas Havelock, Waves due to a sphere making periodic heaving oscillations, *Proc. Roy. Soc. London*, Ser. A231 (1955) 1–7.

[7] A. J. Hermans, A perturbation method for the radiation of surface waves, *J. Eng. Math.*, 6 (1972) 323–330.
A. J. Hermans, A perturbation method for the radiation of short surface waves in three dimensions, *ibid.*, 7 (1973) 75–84.

[8] F. G. Leppington, On the radiation and scattering of short surface waves, I–III, *J. Fluid Mech.*, 56 (1972) 101–119; 59 (1973) 129–146, 147–157.

[9] G. Alker, On the radiation of short surface waves by a heaving, circular cylinder, *J. Eng. Math.*, 9 (1975) 197–205.

[10] C. A. N. Morris, A variational approach to an unsymmetric water-wave scattering problem, *J. Eng. Math.*, 9 (1975) 291–300.

[11] John V. Wehausen and Edmund V. Laitone, Surface waves, in: *Handbuch der Physik*, Vol. 9, Pt. 3, *Fluid Dynamics*, pp. 445–778, Ed. S. Flugge, (1960).

[12] See for example L. S. Rodberg and R. M. Thaler, *The Quantum Theory of Scattering*, Academic Press (New York and London), 1967. The theorem is so called because of the analogy with light passing through a medium. The imaginary part of the index of refraction is related to the absorption coefficient, i.e., the total absorption cross section. See L. Rosenfeld, *Theory of Electrons*, North-Holland Publishing Co. (Amsterdam), 1951.

[13] See for example H. C. van der Hulst, *Light Scattering by Small Particles*, Wiley (New York) 1957.

[14] Philip M. Morse and Herman Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York (1953) p. 887.

[15] *Ibid.*, p. 1466.
[16] A. Erdélyi, *Higher Transcendental Functions*, McGraw-Hill, New York, Vol. 1 (1953) p. 148.
[17] *Ibid.*, p. 164.
[18] Philip M. Morse and Herman Feshbach, *loc. cit.*, p. 1573.
[19] E. O. Steinborn and K. Ruedenberg, Rotation and translation of regular and irregular solid spherical harmonics, in: *Advances in Quantum Chemistry*, Academic Press, New York and London, Vol. 7 (1973) 1–81.
[20] Philip M. Morse and Herman Feshbach, *loc. cit.*, p. 1467; also A. Erdélyi, *loc. cit.*, Vol. 2, p. 57.
[21] Eldon R. Hansen, *A Table of Series and Products*, p. 449 (74.2.24), Prentice-Hall (Englewood, N. J.) 1975; also G. N. Watson, *Theory of Bessel Functions*, Second ed., p. 22, § 2.22(3), Cambridge University Press (London) 1962.
[22] G. N. Watson, *loc. cit.*, p. 79, § 3.71 (9).
[23] *Ibid.*, p. 373, § 12.11 (1).
[24] A. M. J. Davis, Short surface waves due to an oscillating half-immersed sphere, *Mathematika*, 18 (1971) 20–39.
A. M. J. Davis, Short surface waves in the presence of a submerged sphere, *SIAM J. Appl. Math.*, 27 (1974) 464–478.
A. M. J. Davis, Short surface waves in the presence of a submerged circular cylinder, *SIAM J. Appl. Math.*, 27 (1974) 479–491.